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Anti-self-dual Yang–Mills equations on noncommutative space–time

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Abstract

By replacing the ordinary product with the so-called $\star$-product, one can construct an analog of the anti-self-dual Yang–Mills (ASDYM) equations on the noncommutative $\mathbb{R}^4$. Many properties of the ordinary ASDYM equations turn out to be inherited by the $\star$-product ASDYM equation. In particular, the twistorial interpretation of the ordinary ASDYM equations can be extended to the noncommutative $\mathbb{R}^4$, from which one can also derive the fundamental structures for integrability such as a zero-curvature representation, an associated linear system, the Riemann–Hilbert problem, etc. These properties are further preserved under dimensional reduction to the principal chiral field model and Hitchin's Higgs pair equations. However, some structures relying on finite dimensional linear algebra break down in the $\star$-product analogs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The deformed ADHM construction of Nekrasov and Schwarz [1] suggests that the anti-self-dual Yang–Mills (ASDYM) equations will be "integrable" on noncommutative space–times as well. This is also advocated by the work of Kapustin et al. [2] that extends the ordinary twistorial interpretation of the ADHM construction [3,4] to the noncommutative $\mathbb{R}^4$. Since twistor theory is a clue to the integrability of the ordinary ASDYM equations [5], it is natural to expect that the ASDYM equations on the noncommutative $\mathbb{R}^4$, too, will be integrable.

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This issue is also interesting from the point of view of integrable systems of two-dimensional field theories, such as the principal chiral field (PCF) model [6,54] and Hitchin's Higgs pair equations [7]. It is well known that these integrable systems can be derived from the ASDYM equations by dimensional reduction. If a similar reduction procedure works on the noncommutative $\mathbb{R}^4$, it seems likely that the integrability of the four-dimensional system will be inherited by the two-dimensional systems. This four-dimensional point of view can be an alternative approach to recent studies on the PCF and Wess–Zumino–Witten (WZW) models on noncommutative space–times [8–12].

This paper aims to answer these questions. The gauge group is assumed to be $U(N)$ throughout the paper. The ASDYM equations on the noncommutative $\mathbb{R}^4$ are then obtained from the ordinary ASDYM equations by replacing the product of fields in the field equations with the so-called "$\star$-product" (the commutator of which is the Moyal bracket [13]). We shall show that almost all part of the twistorial and integrable structures of the ordinary ASDYM equations can be extended to the noncommutative $\mathbb{R}^4$ by the same substitution rule. What breaks down is the part where tools of finite dimensional linear algebra (determinants, Camer's formula, characteristic polynomials, etc.) are used.

This paper is organized as follows. Section 2 presents the formulation of the $\star$-product ASDYM equations. Section 3 deals with the twistorial and integrable structures of the $\star$-product ASDYM equations. Section 4 is concerned with some implications of the deformed ADHM construction. Section 5 is devoted to two-dimensional reductions. Section 6 is for conclusion.

2. ASDYM equations on noncommutative $\mathbb{R}^4$

2.1. Space–time coordinates

The noncommutative $\mathbb{R}^4$ is characterized by the commutation relations

$$[x_j, x_k] = \mathrm{i}\theta_{jk} \tag{2.1}$$

of the space–time coordinates, where θ_{jk} are real constants. These commutation relations can be extended to the associative $\star$-product

$$f \star g(x) = \exp\left(\sum_{j,k=1}^{4} \frac{\mathrm{i}}{2}\theta_{jk}\partial_{x_j}\partial_{\tilde{x}_k}\right) f(x)g(\tilde{x})\bigg|_{\tilde{x}=x} \tag{2.2}$$

of functions f and g on the space–time.

We now introduce complex coordinates (z_1, z_2) that satisfy commutation relations of the form

$$[z_1, z_2] = -\zeta_{\mathbb{C}}, \qquad [\bar{z}_1, \bar{z}_2] = -\bar{\zeta}_{\mathbb{C}}, \qquad [z_1, \bar{z}_1] + [z_2, \bar{z}_2] = -\zeta_{\mathbb{R}}. \tag{2.3}$$

For instance, $z_1 = x_3 + \mathrm{i}x_4$ and $z_2 = x_1 + \mathrm{i}x_2$ give such complex coordinates after a suitable orthogonal transformation of the real coordinates. The complex constant $\zeta_{\mathbb{C}}$ and the real constant $\zeta_{\mathbb{R}}$ form a three-vector $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ in $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$, and can be rotated to any direction by the $SU(2)$ action

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \tag{2.4}$$

$(|\alpha|^2 + |\beta|^2 = 1)$ on the space–time coordinates.

2.2. ASDYM equations

Let A be an $N \times N$ matrix-valued one-form representing an $U(N)$-connection. Let A_1, A_2, A_3, A_4 denote the components in the real coordinate frame, and $A_{z_1}, A_{z_2}, A_{\bar{z}_1}, A_{\bar{z}_2}$ the components in the complex coordinate frame:

$$A = \sum_{j=1}^{4} A_j \, \mathrm{d}x_j = \sum_{a=1,2} A_{z_a} \, \mathrm{d}z_a + \sum_{a=1,2} A_{\bar{z}_a} \, \mathrm{d}\bar{z}_a. \tag{2.5}$$

The covariant derivatives can be accordingly written as

$$\nabla_{x_j} = \partial_{x_j} + A_j, \qquad \nabla_{z_a} = \partial_{z_a} + A_{z_a}, \qquad \nabla_{\bar{z}_a} = \partial_{\bar{z}_a} + A_{\bar{z}_a}. \tag{2.6}$$

On the noncommutative $\mathbb{R}^4$, the components of the curvature two-form $F = \sum_{j,k} F_{jk} \, \mathrm{d}x_j \wedge \mathrm{d}x_k$ are defined as

$$F_{jk} = \partial_{x_j} A_k - \partial_{x_k} A_j + [A_j, A_k]_\star. \tag{2.7}$$

Note that the usual matrix commutators $[A_j, A_k] = A_j A_k - A_k A_j$ are now replaced by the $\star$-product commutators

$$[A_j, A_k]_\star = A_j \star A_k - A_k \star A_j. \tag{2.8}$$

The components $F_{z_1 z_2}$, $F_{\bar{z}_1 \bar{z}_2}$ and $F_{z_a \bar{z}_b}$ in the complex coordinate frame are similarly written in terms of A_{z_a} and $A_{\bar{z}_a}$. The ASDYM equations in the complex coordinate frame take the neat form

$$F_{z_1 z_2} = 0, \qquad F_{\bar{z}_1 \bar{z}_2} = 0, \qquad F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0. \tag{2.9}$$

2.3. Reduced form of ASDYM equations

It is well known that the ASDYM equations can be converted to a (classical) field theory with a Lagrangian formalism. Actually, two types of such expressions are known. One is Yang's equation [14] (also called the four-dimensional Donaldson–Nair–Schiff equation [15,16,55,56]). Another expression is due to Leznov [17] and Parkes [18]. Both can be extended to the noncommutative space–time as follows.

Let us consider the first equation $F_{z_1 z_2} = 0$ of the ASDYM equations. This is a partial (two-dimensional) flatness condition. In the ordinary (complexified) space–time, this implies that A_{z_1} and A_{z_2} can be expressed as

$$A_{z_1} = h^{-1} \partial_{z_1} h, \qquad A_{z_2} = h^{-1} \partial_{z_2} h \tag{2.10}$$

with an $N \times N$ matrix-valued function h of the space–time coordinates. (Of course, this is, in general, a local expression.) This expression persists to be true on the noncommutative space–time if the ordinary product in the matrix multiplication are replaced by the $\star$-product:

$$A_{z_1} = (h)_\star^{-1} \star \partial_{z_1} h, \qquad A_{z_2} = (h)_\star^{-1} \star \partial_{z_2} h. \tag{2.11}$$

Here $(h)_\star^{-1}$ stands for an inverse with respect to the $\star$-product, namely, $h \star (h)_\star^{-1} = (h)_\star^{-1} \star h = 1$. One can prove this $\star$-product version of Frobenius' theorem in much the same way as a proof in the ordinary space–time.

In order to derive a $\star$-product analog of Yang's equation, we solve another flatness condition $F_{\bar{z}_1,\bar{z}_2} = 0$ in the ASDYM equation as

$$A_{\bar{z}_1} = (k)_\star^{-1} \star \partial_{\bar{z}_1} k, \qquad A_{\bar{z}_2} = (k)_\star^{-1} \star \partial_{\bar{z}_2} k, \tag{2.12}$$

and the "matrix-ratio"

$$g = k \star (h)_\star^{-1} \tag{2.13}$$

of h and k. As we shall show below, this matrix-valued field turns out to obey the field equation

$$\partial_{z_1}((g)_\star^{-1} \star \partial_{\bar{z}_1} g) + \partial_{z_2}((g)_\star^{-1} \star \partial_{\bar{z}_2} g) = 0. \tag{2.14}$$

This gives an analog of Yang's equation on the noncommutative space–time. The Lagrangian formalism in the commutative case can be readily extended to the noncommutative case. Note that the field equation

$$\partial_z((g)_\star^{-1} \star \partial_{\bar{z}} g) + \partial_{\bar{z}}((g)_\star^{-1} \star \partial_z g) = 0 \tag{2.15}$$

of the noncommutative PCF model [10] can be derived by dimensional reduction.

The foregoing noncommutative analog of Yang's equation can be derived by the following trick. Let us consider the finite gauge transformation by h. Two of the four gauge potentials, A_{z_a} $(a = 1, 2)$, are thereby gauged away as

$$\nabla_{z_a} \to h \circ \nabla_{z_a} \circ (h)_\star^{-1} = \partial_{z_a}, \tag{2.16}$$

and the other two are transformed as

$$\begin{aligned} \nabla_{\bar{z}_a} \to h \circ \nabla_{\bar{z}_a} \circ (h)_\star^{-1} &= \partial_{\bar{z}_a} - \partial_{\bar{z}_a} h \star (h)_\star^{-1} + h \star (k)_\star^{-1} \star \partial_{\bar{z}_a} k \star (h)_\star^{-1} \\ &= \partial_{\bar{z}_a} + (g)_\star^{-1} \star \partial_{\bar{z}_a} g. \end{aligned} \tag{2.17}$$

The gauge potentials are now in a *half-flat* gauge in which two of the gauge potentials vanish,

$$A_{z_1} = 0, \qquad A_{z_2} = 0, \tag{2.18}$$

and the other two gauge potentials are written as

$$A_{\bar{z}_a} = (g)_\star^{-1} \star \partial_{\bar{z}_a} g. \tag{2.19}$$

The remaining equation, $F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$ of the $\star$-product ASDYM equations, which now takes the simplified form

$$\partial_{z_1} A_{\bar{z}_1} + \partial_{z_2} A_{\bar{z}_2} = 0, \tag{2.20}$$

gives the $\star$-product analog of Yang's equation.

One can see, from this derivation of Yang's equation, the existence of a field theoretical "dual" of Yang's equation as well. Note that the $\star$-product ASDYM equations in the foregoing half-flat gauge with $A_{z_1} = A_{z_2} = 0$ consist of the two equations

$$\partial_{\bar{z}_1} A_{\bar{z}_2} - \partial_{\bar{z}_2} A_{\bar{z}_1} + [A_{\bar{z}_1}, A_{\bar{z}_2}]_\star = 0, \qquad \partial_{z_1} A_{\bar{z}_1} + \partial_{z_2} A_{\bar{z}_2} = 0. \tag{2.21}$$

If one solves the first equation as a partial flatness condition, as we have seen above, Yang's equation emerges from the second equation. Meanwhile, one can also solve the second equation as

$$A_{\bar{z}_1} = -\partial_{z_2}\phi, \qquad A_{\bar{z}_2} = \partial_{z_1}\phi \tag{2.22}$$

for a matrix-valued potential ϕ. The first equation then takes the form

$$(\partial_{z_1}\partial_{\bar{z}_1} + \partial_{z_2}\partial_{\bar{z}_2})\phi + [\partial_{z_1}\phi, \partial_{z_2}\phi]_\star = 0. \tag{2.23}$$

This is a $\star$-product version of the field equation of Leznov and Parkes.

3. Twistor theory and integrability

3.1. Twistor geometry

Twistor theory encodes various fields on space–time to a geometric structure on another (complex) manifold called the "twistor space" [19,20]. In the case of four-dimensional flat space–time, the twistor space is the three-dimensional complex projective space $\mathbb{P}^3_{\mathbb{C}}$. Roughly speaking, twistor theory is a kind of "tomography", namely, to "scan" the space–time by a three-parameter family of two-dimensional surfaces ("twistor surfaces") $S(\xi)$ labeled by the point ξ of the twistor space. We review the essence of twistor geometry in the following.

To define the twistor surfaces, however, the real (Euclidean) space–time $\mathbb{R}^4$ has to be extended to the complexified space–time $\mathbb{C}^4$, in which $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ are *independent* complex coordinates. The twistor surfaces in the complexified space–time $\mathbb{C}^4$ are labeled by three parameters (λ, u_1, u_2), and defined by the equations

$$z_1 - \lambda\bar{z}_2 = u_1, \qquad z_2 + \lambda\bar{z}_1 = u_2. \tag{3.1}$$

The parameters (λ, u_1, u_2) are local coordinates on a coordinate patch of the whole twistor space $\mathbb{P}^3_{\mathbb{C}}$. Furthermore, λ turns out to play the role of the "spectral parameter" in the theory of integrable systems.

Various real space–times, such as the Minkowski space–time and the space–time with (2, 2) signature, are embedded in the complexified space–time $\mathbb{C}^4$ as "real slices". Although the twistor surface $S(\lambda, u_1, u_2)$ intersects with the Euclidean space–time at most at a point, the intersection with the Minkowski space–time is a null line, and the intersection with the $2+2$ space–time is a totally null surface (i.e., the inner product of any two tangent vectors vanish).

The twistor space $\mathbb{P}^3_{\mathbb{C}}$ itself appears in the description of a compactified space–time, such as the one-point compactification $S^4 = \mathbb{R}^4 \cup \{\infty\}$ of the Euclidean space–time. Let us introduce the complex Grassmann variety

$$Gr_{\mathbb{C}}(2,4) = \{V_2 | V_2 \subset \mathbb{C}^4, \dim V_2 = 2\} \tag{3.2}$$

of vector subspaces of $\mathbb{C}^4$ and the flag variety

$$Fl_{\mathbb{C}}(1,2,4) = \{(V_1, V_2) | V_1 \subset V_2 \subset \mathbb{C}^4, \dim V_1 = 1, \dim V_2 = 2\} \tag{3.3}$$

of pairs of nested vector subspaces of $\mathbb{C}^4$. The Grassmann variety is a natural complexification of S^4. Twistor geometry connects these compact (and complexified) space–times with the twistor space $\mathbb{P}^3_{\mathbb{C}}$ by the "Klein correspondence"

$$Gr_{\mathbb{C}}(2,4) \xleftarrow{p_2} Fl_{\mathbb{C}}(1,2,4) \xrightarrow{p_1} \mathbb{P}^3_{\mathbb{C}}, \tag{3.4}$$

where the projections p_1 and p_2 send the flag (V_1, V_2) to $V_1 \in \mathbb{P}^3_{\mathbb{C}}$ and $V_2 \in Gr_{\mathbb{C}}(2,4)$, respectively. The subset $S(\xi) = p_2(p_1^{-1}(\xi))$ is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ and gives a compactification of the foregoing twistor surface $S(\lambda, u_1, u_2)$ in $\mathbb{C}^4$. Similarly, the subset $L(x) = p_1(p_2^{-1}(x))$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and plays a key role in *decoding* the twistorial data.

The uncompactified space–time $\mathbb{R}^4$ (or, rather, its complexification $\mathbb{C}^4$) can be described by the open twistor space

$$\mathcal{T} = \mathbb{P}^3_{\mathbb{C}} \setminus \mathbb{P}^1_{\mathbb{C}} \tag{3.5}$$

with a line $\mathbb{P}^1_{\mathbb{C}}$ deleted. It is rather this twistor space that we mostly consider in the following. This open twistor space has the projection

$$\pi : \mathcal{T} \to \mathbb{P}^1_{\mathbb{C}}, \qquad \xi = [\xi_0 : \xi_1 : \xi_2 : \xi_3] \mapsto [\xi_0 : \xi_1], \tag{3.6}$$

and is covered by the two standard coordinate patches $\mathcal{U} = \{\xi_0 \neq 0\}$ and $\hat{\mathcal{U}} = \{\xi_1 \neq 0\}$. The deleted line $\mathbb{P}^1_{\mathbb{C}}$ is the locus where $\xi_0 = \xi_1 = 0$. The three parameters (λ, u_1, u_2) can be identified with the standard local coordinates on $\mathcal{U}$:

$$\lambda = \frac{\xi_1}{\xi_0}, \qquad u_1 = \frac{\xi_2}{\xi_0}, \qquad u_2 = \frac{\xi_3}{\xi_0}. \tag{3.7}$$

Thus, in particular, λ is an affine coordinate of the base, and u_1 and u_2 are coordinates along the fibers.

3.2. Flatness on twistor surfaces

The three members of the ASDYM equations can be combined to a single equation of the form

$$F(\partial_{\bar{z}_1} - \lambda\partial_{z_2}, \partial_{\bar{z}_2} + \lambda\partial_{z_1}) = F_{z_1 z_2} - \lambda(F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2}) + \lambda^2 F_{\bar{z}_1\bar{z}_2} = 0. \tag{3.8}$$

Here $F(v, v')$ stands for the contraction of F by two vector fields v, v'. Since the two vector fields $\partial_{\bar{z}_1} - \lambda\partial_{z_2}, \partial_{\bar{z}_2} + \lambda\partial_{z_1}$ on the left-hand side span the tangent planes of the twistor surface $S(\lambda, u_1, u_2)$, the foregoing equation means the flatness

$$F|_{S(\lambda,u_1,u_2)} = 0 \tag{3.9}$$

of the connection on all twistor surfaces.

Frobenius' theorem connects this flatness (or "zero-curvature") condition with the integrability of the linear system [21–23]

$$(\nabla_{\bar{z}_1} - \lambda\nabla_{z_2})\Psi(\lambda) = 0, \qquad (\nabla_{\bar{z}_2} + \lambda\nabla_{z_1})\Psi(\lambda) = 0, \tag{3.10}$$

where $\Psi(\lambda)$ is a vector- or matrix-valued unknown function (which, of course, depends on the space–time coordinates as well). Having this linear system, one can now apply a number of techniques for integrable systems to the ASDYM equations [5].

Note that the first two equations of the ASDYM equations (from which h and k were derived) correspond to the flatness on the twistor surfaces with $\lambda = 0$ and $\lambda = \infty$. Accordingly, one can choose two matrix-valued solutions $\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ of (3.10) to be such that

$$\Psi(0) = h, \qquad \hat{\Psi}(\infty) = k. \tag{3.11}$$

In other words, $\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ are one-parameter deformations of h and k. Moreover, the Laurent expansion

$$\Psi(\lambda) = h + \sum_{n=1}^{\infty} w_n\lambda^n, \qquad \hat{\Psi}(\lambda) = k + \sum_{n=1}^{\infty} \hat{w}_n\lambda^{-n} \tag{3.12}$$

of these solutions of (3.10) are related to two infinite series of nonlocal conservation laws [22,23]. There is no reason that these two solutions coincide. They rather give a pair that arise in the so-called Riemann–Hilbert problem. We now turn to this issue.

3.3. Vector bundle and Riemann–Hilbert problem

The twistor transformation [24–27] encodes a solution of the ASDYM equations to a holomorphic vector bundle $\mathcal{E}$ over the twistor space. Given a solution of the ASDYM equations, one can consider an associated rank N vector bundle E over the space–time with an induced connection. This connection is flat on each twistor surface $S(\xi)$. The fiber $\mathcal{E}_\xi$ of the bundle $\mathcal{E}$ at a point ξ of the twistor space is, by definition, the vector space of flat sections of $E|_{S(\xi)}$. In a down-to-earth language, the fiber $\mathcal{E}_\xi$ is the vector space of E-valued solutions of linear system (3.10) restricted to $S(\lambda, u_1, u_2)$.

This bundle $\mathcal{E}$ need not be defined over the whole twistor space. If the solution of the ASDYM equations is defined in a small neighborhood of a space–time point x, the bundle $\mathcal{E}$ is accordingly defined only in a neighborhood of the line $L(x) = \{\xi \in \mathbb{P}^3_{\mathbb{C}} | x \in S(\xi)\}$. Instanton solutions are global solutions that give rise to a globally defined vector bundle on the whole twistor space.

The holomorphic vector bundle $\mathcal{E}$ has the special property that the restriction $\mathcal{E}|_{L(x)}$ to the line $L(x) \simeq \mathbb{P}^1_{\mathbb{C}}$ is holomorphically trivial for any space–time point x in the domain where the gauge potentials are defined. This property of $\mathcal{E}$ plays a key role in the *inverse* transformation, namely, to reproduce the solution of the ASDYM equation from the vector bundle $\mathcal{E}$.

It is here that the notion of Riemann–Hilbert problem emerges. Let us recall that any holomorphic vector bundle over $\mathbb{P}^1_{\mathbb{C}}$ can be represented by the "patching function" $p(\lambda)$ on the intersection $D \cap \hat{D}$ of two affine coordinate patches $\{D, \hat{D}\}$ of $\mathbb{P}^1_{\mathbb{C}}$. The patching function $p(\lambda)$ is a $GL(N, \mathbb{C})$-valued holomorphic function. If the vector bundle is holomorphically trivial, the patching function can be expressed as

$$p(\lambda) = \hat{\Psi}(\lambda)^{-1} \Psi(\lambda), \tag{3.13}$$

where $\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ are $GL(N, \mathbb{C})$-valued holomorphic functions of λ on D and $\hat{D}$, respectively. Finding such a pair of matrix-valued functions to the given data $p(\lambda)$ is a kind of Riemann–Hilbert problem (also called the "splitting" problem in the terminology of Ward).

The patching function $p(\lambda)$ is determined by a patching function of the vector bundle $\mathcal{E}$ itself. As already remarked, the twistor space $\mathcal{T} = \mathbb{P}^3_{\mathbb{C}} \setminus \mathbb{P}^1_{\mathbb{C}}$ is covered by the two coordinate patches $\mathcal{U}$ and $\hat{\mathcal{U}}$. The vector bundle $\mathcal{E}$ is described by a $GL(N, \mathbb{C})$-valued function $P(\lambda, u_1, u_2)$ that glues together the rank-N trivial bundles over $\mathcal{U}$ and $\hat{\mathcal{U}}$. Its restriction on the line $L(x)$ is nothing but the patching function $p(\lambda)$ of $\mathcal{E}|_{L(x)}$:

$$p(\lambda) = P(\lambda, z_1 - \lambda \bar{z}_2, z_2 + \lambda \bar{z}_1). \tag{3.14}$$

In particular, the patching function $p(\lambda)$ turns out to obey the linear differential equations

$$(\partial_{\bar{z}_1} - \lambda \partial_{z_2}) p(\lambda) = 0, \qquad (\partial_{\bar{z}_2} + \lambda \partial_{z_1}) p(\lambda) = 0. \tag{3.15}$$

(Note that this is a rather simplified setup. If the solution is defined in a general domain of space–time, we need a more refined cohomological language — see Ivanova's review [28] and references cited therein.)

Given such a patching function, one can prove that the Riemann–Hilbert problem indeed solves the ASDYM equations. We shall review this proof later on in the framework of the noncommutative space–time. The converse is also true. Namely, if $\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ are a pair of *arbitrary* solutions of (3.10), its matrix ratio $\hat{\Psi}(\lambda)^{-1}\Psi(\lambda)$ satisfies Eq. (3.15). This can be confirmed by direct calculations.

Solving the Riemann–Hilbert problem *explicitly* is usually very difficult. Explicit solutions are known for special cases only. The so-called "Ward Ansatz" (or "Atiyah–Ward

Ansatz") solutions [25,29,30] provide such an example. The corresponding Riemann–Hilbert problem can be solved by linear algebra.

The existence of a large set of hidden symmetries of the ASDYM equations [31,32] can be explained by the Riemann–Hilbert problem [33,34]. Those symmetries are generated by the left and right action

$$p(\lambda) \mapsto g_{\mathrm{L}}(\lambda) p(\lambda) g_{\mathrm{R}}(\lambda)^{-1} \tag{3.16}$$

of $GL(N, \mathbb{C})$-valued functions $g_{\mathrm{L}}(\lambda)$ and $g_{\mathrm{R}}(\lambda)$ of $(\lambda, z_1 - \lambda\bar{z}_2, z_2 + \lambda\bar{z}_1)$. The infinitesimal form of these symmetries can be determined explicitly and reproduces the previously known results [35,57]. For subsequent progress on finite transformations, see Popov's paper [36].

3.4. Integrability of $\star$-product ASDYM equations

Having reviewed the twistorial and integrable structures of the ordinary ASDYM equations, we now turn to the $\star$-product ASDYM equations.

The geometric setup of twistor theory can be extended to the noncommutative space–time rather straightforward. To see this, let us notice that the commutation relations of the complex coordinates $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ can be rewritten as

$$[z_1 - \lambda\bar{z}_2, z_2 + \lambda\bar{z}_1] = -\zeta_{\mathbb{C}} - \lambda\zeta_{\mathbb{R}} + \lambda^2\bar{\zeta}_{\mathbb{C}}. \tag{3.17}$$

The linear combinations of the space–time coordinates on the left-hand side are exactly those in the definition of the twistor surface $S(\lambda, u_1, u_2)$. Accordingly, whereas λ persists to be a commutative coordinate, the coordinates u_1 and u_2 of the fibers of $\pi : \mathcal{T} \to \mathbb{P}^1_{\mathbb{C}}$ turn out to have to obey the commutation relation

$$[u_1, u_2] = -\zeta_{\mathbb{C}} - \lambda\zeta_{\mathbb{R}} + \lambda^2\bar{\zeta}_{\mathbb{C}}. \tag{3.18}$$

Thus the twistor space, like the space–time, becomes a noncommutative manifold. This will be an alternative interpretation of the results of Kapustin et al. [2].

The linear system for $\Psi(\lambda)$ is now replaced by the $\star$-product version

$$\begin{aligned}
&(\nabla_{\bar{z}_1} - \lambda\nabla_{z_2}) \star \Psi(\lambda) = (\partial_{\bar{z}_1} - \lambda\partial_{z_2})\Psi(\lambda) + (A_{\bar{z}_1} - \lambda A_{z_2}) \star \Psi(\lambda) = 0,\\
&(\nabla_{\bar{z}_2} + \lambda\nabla_{z_1}) \star \Psi(\lambda) = (\partial_{\bar{z}_2} + \lambda\nabla_{z_1})\Psi(\lambda) + (A_{\bar{z}_2} + \lambda A_{z_1}) \star \Psi(\lambda) = 0.
\end{aligned} \tag{3.19}$$

Although the notion of vector bundles on the noncommutative twistor space is complicated [2], the Riemann–Hilbert problem itself remains intact except that the product $\hat{\Psi}(\lambda)^{-1}\Psi(\lambda)$ is replaced by the $\star$-product:

$$p(\lambda) = (\hat{\Psi}(\lambda))_\star^{-1} \star \Psi(\lambda). \tag{3.20}$$

The patching function $p(\lambda)$ is required to satisfy the same linear differential equations as (3.15), or, equivalently, to be of the form $P(\lambda, z_1 - \lambda\bar{z}_2, z_2 + \lambda\bar{z}_1)$.

Let us confirm that this Riemann–Hilbert problem indeed solves the $\star$-product ASDYM equations. The reasoning is fully parallel to the ordinary ASDYM equations. We first note that (3.15) implies the equations

$$(\partial_{\bar z_1} - \lambda\partial_{z_2})\Psi(\lambda) \star (\Psi(\lambda))_\star^{-1} = (\partial_{\bar z_1} - \lambda\partial_{z_2})\hat\Psi(\lambda) \star (\hat\Psi(\lambda))_\star^{-1},$$
$$(\partial_{\bar z_2} + \lambda\partial_{z_1})\Psi(\lambda) \star (\Psi(\lambda))_\star^{-1} = (\partial_{\bar z_2} + \lambda\partial_{z_1})\hat\Psi(\lambda) \star (\hat\Psi(\lambda))_\star^{-1}. \tag{3.21}$$

Since D and $\hat D$ cover the whole Riemann sphere, both hand sides of these equations define a matrix-valued meromorphic function with the only possible poles being at $\lambda = \infty$ and of the first order. By Liouville's theorem, they are a linear function of λ with matrix coefficients. Let us express these linear functions as $-A_{\bar z_1} + \lambda A_{z_2}$ and $-A_{\bar z_2} - \lambda A_{z_1}$. The coefficients $A_{z_1}, A_{z_2}, A_{\bar z_1}, A_{\bar z_2}$ are to be identified with the gauge potentials. Thus $\Psi(\lambda)$ and $\hat\Psi(\lambda)$ turn out to satisfy (3.19), from which the $\star$-product ASDYM equations are derived.

The other part of the forgoing discussion, too, can be mostly extended to the noncommutative space–time. For instance, hidden symmetries are again generated by the action

$$p(\lambda) \mapsto g_{\mathrm{L}}(\lambda) \star p(\lambda) \star g_{\mathrm{R}}(\lambda)^{-1} \tag{3.22}$$

of $GL(N, \mathbb{C})$-valued functions $g_{\mathrm{L}}(\lambda)$ and $g_{\mathrm{R}}(\lambda)$ of $(\lambda, z_1 - \lambda\bar z_2, z_2 + \lambda\bar z_1)$. The associated infinitesimal symmetries take the same form as those for the ordinary ASDYM equations (with, of course, the product of space–time functions being replaced by the $\star$-product).

An essential difference can be seen in the places where finite dimensional linear algebra is used. A typical example is the Ward Ansatz. In the noncommutative framework, such a linear algebraic structure has to be replaced by an infinite dimensional counterpart. As for the Ward Ansatz, for instance, we do not know how to extend it to the $\star$-product ASDYM equations.

4. Deformed ADHM construction

4.1. How to deform ADHM construction

The ordinary ADHM construction [3,4] of an $U(N)$-instanton solution with instanton number k is based on the $2k \times (2k + N)$ matrix-valued function

$$\Delta(z) = \begin{pmatrix} B_1 + z_1 1 & B_2 + z_2 1 & I \\ -B_2^\dagger - \bar z_2 1 & B_1^\dagger + \bar z_1 1 & J^\dagger \end{pmatrix} \tag{4.1}$$

of $z = (z_1, z_2, \bar z_1, \bar z_2)$. Here B_1 and B_2 are $k\times k$ matrices, I a $k\times N$ matrix, J an $N\times k$ matrix, and $B_1^\dagger, B_2^\dagger, I^\dagger, J^\dagger$ their Hermitian conjugate. Assuming a nondegeneracy condition, one can construct a $(2k + N) \times N$ matrix $v(z)$ that satisfies the equations

$$\Delta(z)v(z) = 0, \qquad v(z)^\dagger v(z) = 1. \tag{4.2}$$

If the so-called ADHM equations

$$[B_1, B_2] + IJ = 0, \qquad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{4.3}$$

are satisfied, the gauge potentials defined by

$$A = v(z)^\dagger\, \mathrm{d}v(z) \tag{4.4}$$

give a solution (instanton solution) of the ASDYM equations.

As Nekrasov and Schwarz [1] pointed out, the instanton solutions of the ASDYM equations on the noncommutative $\mathbb{R}^4$ can be obtained by deforming the ADHM equations as

$$[B_1, B_2] + IJ = \zeta_{\mathbb{C}} 1, \qquad [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta_{\mathbb{R}} 1. \tag{4.5}$$

The connection form is now given by the $\star$-product

$$A = v(z)^\dagger \star \mathrm{d}v(z). \tag{4.6}$$

4.2. Solution of Riemann–Hilbert problem

We here present, as an application of the ADHM construction, an explicit construction of the solution of the Riemann–Hilbert problem for the instanton solutions. This is based on the work of Corrigan et al. [37] on the Dirac equation with the instanton gauge potentials.

According to one of their results, the parallel translation (i.e., the “Wilson line operator”)

$$w(z, z') = P - \exp\left(\int_{z'}^{z} A\right) \tag{4.7}$$

between two points z, z' on the same twistor surface $S(\lambda, u_1, u_2)$ is given by the simple formula

$$w(z, z') = v(z)^\dagger v(z'). \tag{4.8}$$

Consequently, this matrix obeys the group law

$$w(z, z')w(z', z'') = w(z, z'') \tag{4.9}$$

for any triple z, z', z'' of points on $S(\lambda, u_1, u_2)$.

Let us apply this group law to the special points

$$z^\infty(\lambda) = (z_1 - \lambda\bar{z}_2, z_2 + \lambda\bar{z}_1, 0, 0),$$
$$z^0(\lambda) = (0, 0, \bar{z}_1 + \lambda^{-1}z_2, \bar{z}_2 - \lambda^{-1}z_1) \tag{4.10}$$

that are on the same twistor surface as $z = (z_1, z_2, \bar{z}_1, \bar{z}_2)$. Accordingly, we have the relation

$$w(z^\infty(\lambda), z^0(\lambda)) = w(z, z^\infty(\lambda))^{-1} w(z, z^0(\lambda)). \tag{4.11}$$

This relation is exactly the Riemann–Hilbert problem with the patching function

$$p(\lambda) = w(z^\infty(\lambda), z^0(\lambda)) \tag{4.12}$$

for which we thus obtain the explicit solution

$$\Psi(\lambda) = w(z, z^0(\lambda)), \qquad \hat{\Psi}(\lambda) = w(z, z^\infty(\lambda)). \tag{4.13}$$

This construction carries over to the noncommutative case if the ordinary matrix products therein are replaced by the $\star$-product. The parallel translation along the twistor surface is given by the $\star$-product

$$w(z, z') = v(z)^\dagger \star v(z'), \tag{4.14}$$

and the foregoing expressions of $g(\lambda)$, $\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ remain valid.

4.3. Remarks on ADHM equations

It is well known that the left-hand side of the ADHM equations, i.e.,

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ, \qquad \mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \tag{4.15}$$

are a pair of moment maps for the hyper-Kähler quotient construction [38] of the moduli space of (both undeformed and deformed) ADHM instantons. In particular, the pair $(\mu_{\mathbb{C}}, \mu_{\mathbb{R}})$ transforms just like the three-vector $(\zeta_{\mathbb{C}}, \zeta_{\mathbb{R}})$ under the $SU(2)$ rotation (2.4) of space–time coordinates. Therefore it is natural to combine the three moment maps $\mu_{\mathbb{C}}, \mu_{\mathbb{R}}, \mu_{\mathbb{R}}^\dagger$ to the one-parameter family

$$\mu(\lambda) = \mu_{\mathbb{C}} + \lambda\mu_{\mathbb{R}} - \lambda^2\mu_{\mathbb{C}}^\dagger = [B_1 - \lambda B_2^\dagger, B_2 + \lambda B_1^\dagger] + (I - \lambda J^\dagger)(J + \lambda I^\dagger) \tag{4.16}$$

of moment maps. The $SU(2)$ action is now represented by fractional transformations of λ:

$$\lambda \mapsto \frac{-\beta + \alpha\lambda}{\bar{\alpha} + \bar{\beta}\lambda}. \tag{4.17}$$

A "pencil" of moment maps of this type generally appears in the quotient construction of the twistor space associated with a hyper-Kähler quotient [38]. The twistor space $\mathcal{Z}$ of a hyper-Kähler manifold is fibered over $\mathbb{P}^1_{\mathbb{C}}$ by a map $\pi : \mathcal{Z} \to \mathbb{P}^1_{\mathbb{C}}$, and each fiber $\pi^{-1}(\lambda)$ is a complex symplectic manifold. The moment map $\mu(\lambda)$ is used to make the symplectic quotient of $\pi^{-1}(\lambda)$. Roughly speaking, this fiberwise symplectic quotient of $\mathcal{Z}$ gives the twistor space for the hyper-Kähler quotient.

This pencil of moment maps is also interesting in the context of finite dimensional integrable systems. Following, Gorsky et al. [39], let us introduce the symplectic form

$$\Omega = \mathrm{Tr}(\mathrm{d}B_1 \wedge \mathrm{d}B_2 + \mathrm{d}I \wedge \mathrm{d}J) \tag{4.18}$$

on the space of the quadruples (B_1, B_2, I, J). As they pointed out, $\mu_{\mathbb{C}}$ may be interpreted as the moment map of the action

$$(B_1, B_2, I, J) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \tag{4.19}$$

of $G = GL(N, \mathbb{C})$, and the reduced phase space (actually, with I and J being further constrained to a special G-orbit) has the structure of an integrable system with the Poisson-commutative Hamiltonians $\mathrm{Tr}B_2^\ell$, $\ell = 1, \ldots, N$. If k is equal to 1 and B_1 and B_2 are restricted to Hermitian matrices, this integrable system reduces to the rational Calogero–Moser system; the case for $k > 1$ is related to a generalized Calogero–Moser system [40]. Now, what occurs if one repeats the same construction for the pencil $\mu(\lambda)$ of moment maps? Note that the symplectic form, too, has to be deformed as

$$\Omega(\lambda) = \mathrm{Tr}(\mathrm{d}(B_1 - \lambda B_2^\dagger) \wedge \mathrm{d}(B_2 + \lambda B_1^\dagger) + \mathrm{d}(I - \lambda J^\dagger) \wedge \mathrm{d}(J + \lambda I^\dagger)). \tag{4.20}$$

Upon taking the symplectic quotient, a one-parameter family of integrable systems will emerge. In fact, $\Omega(\lambda)$ is exactly the symplectic form of the fiber $\pi^{-1}(\lambda)$ of the twistor space before taking the quotient. Thus the phase space of the aforementioned one-parameter family of integrable systems turns out to be nothing but the fibers $\pi^{-1}(\lambda)$ of the twistor space of the instanton moduli space.

5. Two-dimensional reductions

5.1. PCF model and Hitchin's equations

We here examine the PCF model and Hitchin's Higgs pair equations as dimensional reductions of the ASDYM equations.

The PCF model can be derived by letting

$$\nabla_{\bar{z}_1} \to \partial_z + A_z, \quad \nabla_{z_2} \to \partial_z, \qquad \nabla_{\bar{z}_2} \to \partial_{\bar{z}} + A_{\bar{z}}, \quad \nabla_{z_1} \to \partial_{\bar{z}} \tag{5.1}$$

under the gauge $A_{z_1} = A_{z_2} = 0$. The associated linear system reads

$$((1-\lambda)\partial_z - A_z)\Psi(\lambda) = 0, \qquad ((1+\lambda)\partial_{\bar{z}} + A_{\bar{z}})\Psi(\lambda) = 0. \tag{5.2}$$

On the noncommutative space–time, z and $\bar{z}$ are assumed to obey the commutation relation

$$[z, \bar{z}] = -\zeta 1 \tag{5.3}$$

for a real constant ζ, and the linear system is replaced by the $\star$-product analog

$$((1-\lambda)\partial_z - A_z) \star \Psi(\lambda) = 0, \qquad ((1+\lambda)\partial_{\bar{z}} + A_{\bar{z}}) \star \Psi(\lambda) = 0. \tag{5.4}$$

Conservation laws, infinitesimal symmetries, the Riemann–Hilbert problem, etc. [1] can be extended to the $\star$-product PCF model straightforward.

Hitchin's Higgs pair equations

$$F_{z\bar{z}} = [\Phi, \Phi^{\dagger}], \qquad \nabla_{\bar{z}}\Phi = 0, \qquad \nabla_z \Phi^{\dagger} = 0 \tag{5.5}$$

can be derived from the ASDYM equations by first exchanging $z_2 \leftrightarrow \bar{z}_2$ (which interchanges anti-self-duality and self-duality), then reducing

$$\nabla_{\bar{z}_2} \to \nabla_z, \quad \nabla_{z_1} \to \Phi, \qquad \nabla_{z_2} \to \nabla_{\bar{z}}, \quad \nabla_{\bar{z}_1} \to \Phi^{\dagger}, \tag{5.6}$$

while letting $\partial_{z_1} \to 0$ and $\partial_{\bar{z}_1} \to 0$. The associated linear system can be written as

$$(\nabla_z + \lambda\Phi)\Psi(\lambda) = 0, \qquad (\nabla_{\bar{z}} - \lambda^{-1}\Phi^{\dagger})\Psi(\lambda) = 0. \tag{5.7}$$

A natural $\star$-product analog of these equations are, of course,

$$F_{z\bar{z}} = [\Phi, \Phi^{\dagger}]_{\star}, \qquad \nabla_{\bar{z}} \star \Phi = 0, \qquad \nabla_z \star \Phi^{\dagger} = 0, \tag{5.8}$$

$$(\nabla_z + \lambda\Phi) \star \Psi(\lambda) = 0, \qquad (\nabla_{\bar{z}} - \lambda^{-1}\Phi^{\dagger}) \star \Psi(\lambda) = 0. \tag{5.9}$$

[1] A detailed exposition of these issues, along with a large list of references, can be found in [41].

5.2. Some more remarks on Hitchin's equations

Hitchin's equations are formulated on any compact Riemann surface [7]. If the genus of the Riemann surface is greater than one, the moduli space of "stable" Higgs pairs is a smooth (but noncompact) symplectic manifold with the structure of an "algebraically integrable Hamiltonian system" [42]. (This fact is further extended to punctured Riemann surfaces including tori.) The "spectral curve" $\det(\Phi - \zeta 1) = 0$ plays a central role therein.

What about the $\star$-product analog of Hitchin's equations? Unfortunately, we do not know if the moduli space of solutions has any structure of an integrable system, because, first of all, the notion of determinant (hence, of spectral curve) ceases to exist. This is a place where a linear algebraic structure breaks down again. One can nevertheless expect that some yet unknown mechanism might give rise to an integrable structure in the moduli space solutions. This issue will be closely related to the notion of "noncommutative Riemann surfaces" that has been pursued by Bertoldi et al. [43].

Let us finally mention that Hitchin's equations are also related to a class of conformal field theories — e.g., the (nonaffine) Toda field theories [44] and W-gravity [45,46]. The associated $\star$-product analogs will be interesting from the point of view of the Chern–Simons and WZW models on noncommutative spaces [8–12]. Note, however, that the naive substitution prescription $e^{\alpha\cdot\phi} \to (e^{\alpha\cdot\phi})_\star$ in the Toda field theories does not lead to an integrable system. A correct integrable deformation is the so-called "non-Abelian Toda field theory", which does not take such an exponential form.

6. Conclusion

We have shown that many properties of the ASDYM equations are inherited by its analog on the noncommutative $\mathbb{R}^4$. After all, the rule of game is quite simple — just to replace the ordinary product by the $\star$-product. This rather naive prescription has turned out to fit surprisingly well into the twistorial and integrable structures of the ASDYM equations. Moreover, these structures are preserved under dimensional reduction to the PCF model and Hitchin's Higgs pair equations. However, linear algebraic structures, such as the Ward Ansatz solutions, mostly loose its meaning in the noncommutative space–times.

We have also pointed out a few interesting structures in the ADHM construction. These structures deserve to be studied in more detail.

Another important issue, which we have not addressed in this paper, is that of the Nahm equations. The Nahm construction of BPS monopoles [47–49] has been extended to a noncommutative space–time [50], in which a $\star$-product analog of the Nahm equations is used. The $\star$-product Nahm equations have been independently studied in the context of the **M**-theory as well [51–53].

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